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A Constructive Description of Ground States and Gibbs Measures for Ising Model with Two-Step Interactions on Cayley Tree

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We consider the Ising model with (competing) two-step interactions and spin values ± 1, on a Cayley tree of order $k \geq 1$. We constructively describe ground states and verify the Peierls condition for the model. We define notion of a contour for the model on the Cayley tree. Using a contour argument we show the existence of two different Gibbs measures.

KEY WORDS: Cayley tree; Configuration; Ising model; Ground state; Gibbs measure.

1. INTRODUCTION

One of the key problems related to the spin models is the description of the set of Gibbs measures. This problem has a good connection with the problem of the description the set of ground states. Because the phase diagram of Gibbs measures (see [11,20] for details) is close to the phase diagram of the ground states for sufficiently small temperatures.

The ground states for models on the cubic lattice Z^d were studied in many works (see e.g. [7,9,10,16,17]).

The Ising model, with two values of spin ± 1 was considered in [15,21] and became actively researched in the 1990's and afterwards (see for example [1–4,8,13,14,18]).

In the paper we consider an Ising model on a Cayley tree with competing interactions. The goal of the paper is to study of (periodic and non periodic) ground states and to verify the Peierls condition for the model. Using the ground

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states we also will define a notion of contours which allows us to develop a contour argument (Pirogov-Sinai theory) on the Cayley tree. In order to describe an infinite set of ground states we use a construction, which we will develop here.

In [19] a contour argument for q- component models (with nearest-neighbor interaction) on Cayley tree was developed. This paper can be considered as a continuation of the paper[19].

2. DEFINITIONS

2.1. The Cayley Tree

The Cayley tree Γ^k (See [1]) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k, L is the set of edges of Γ^k and i is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called *nearest neighboring vertices*, and we write $l = \langle x, y \rangle$.

The distance $d(x, y), x, y \in V$ on the Cayley tree is defined by the formula

$$d(x, y) = \min\{d | \exists x = x_0, x_1, \ldots, x_{d-1}, x_d = y \in V \text{such that} \langle x_0, x_1 \rangle, \ldots, \langle x_{d-1}, x_d \rangle\}.$$

For the fixed $x^0 \in V$ we set $W_n = \{x \in V \mid d(x, x^0) = n\}$,

$$V_n = \{x \in V \mid d(x, x^0) \leq n\}, \quad L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\}. \tag{1}$$

Denote $|x| = d(x, x^0), x \in V$.

A collection of the pairs $\langle x, x_1 \rangle$, ..., $\langle x_{d-1}, y \rangle$ is called a path from x to y and we write $\pi(x, y)$. We write $x < y$ if the path from x^0 to y goes through x.

It is known (see [8]) that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k+1$ cyclic groups $\{e, a_i\}, i = 1, \ldots, k+1$ of the second order (i.e. $a_i^2 = e, a_i^{-1} = a_i$) with generators $a_1, a_2, \ldots, a_{k+1}$.

Let us define a graph structure on G_k as follows. Vertices which correspond to the "words" $g, h \in G_k$ are called nearest neighbors if either $g = ha_i$ or $h = ga_j$ for some i or j. The graph thus defined is a Cayley tree of order k.

For $g_0 \in G_k$ a left (resp. right) transformation shift on G_k is defined by

$$T_{g_0}h = g_0 h \,(\text{resp. } T_{g_0}h = hg_0,) \quad \forall h \in G_k.$$

It is easy to see that the set of all left (resp. right) shifts on G_k is isomorphic to G_k.

2.2. The Model

We consider models where the spin takes values in the set $\Phi = \{-1, 1\}$. For $A \subseteq V$ a spin *configuration* σ_A on A is defined as a function $x \in A \to \sigma_A(x) \in \Phi$; the set of all configurations coincides with $\Omega_A = \Phi^A$. We denote $\Omega = \Omega_V$ and $\sigma = \sigma_V$. Also put $-\sigma_A = \{-\sigma_A(x), x \in A\}$. We define a *periodic configuration* as a configuration $\sigma \in \Omega$ which is invariant under a subgroup of shifts $G_k^* \subset G_k$ of finite index.

More precisely, a configuration $\sigma \in \Omega$ is called G_k^*-periodic if $\sigma(yx) = \sigma(x)$ for any $x \in G_k$ and $y \in G_k^*$.

For a given periodic configuration the index of the subgroup is called the *period of the configuration*. A configuration that is invariant with respect to all shifts is called *translational-invariant*.

The Hamiltonian of the Ising model with competing interactions has the form

$$H(\sigma) = J_1 \sum_{\langle x,y\rangle} \sigma(x)\sigma(y) + J_2 \sum_{x,y\in V:d(x,y)=2} \sigma(x)\sigma(y) \tag{2}$$

where $J_1, J_2 \in R$ are coupling constants and $\sigma \in \Omega$.

3. GROUND STATES

For a pair of configurations σ and φ that coincide almost everywhere, i.e. everywhere except for a finite number of positions, we consider a relative Hamiltonian $H(\sigma, \varphi)$, the difference between the energies of the configurations σ, φ of the form

$$H(\sigma, \varphi) = J_1 \sum_{\langle x,y\rangle} (\sigma(x)\sigma(y) - \varphi(x)\varphi(y))$$

$$+ J_2 \sum_{x,y\in V:d(x,y)=2} (\sigma(x)\sigma(y) - \varphi(x)\varphi(y)), \tag{3}$$

where $J = (J_1, J_2) \in R^2$ is an arbitrary fixed parameter.

Let M be the set of unit balls with vertices in V. We call the restriction of a configuration σ to the ball $b \in M$ a *bounded configuration* σ_b.

Define the energy of a ball b for configuration σ by

$$U(\sigma_b) \equiv U(\sigma_b, J) = \frac{1}{2} J_1 \sum_{\langle x,y\rangle, x,y\in b} \sigma(x)\sigma(y)$$

$$+ J_2 \sum_{x,y\in b:\ d(x,y)=2} \sigma(x)\sigma(y), \tag{4}$$

where $J = (J_1, J_2) \in R^2$.

We shall say that two bounded configurations σ_b and $\sigma'_{b'}$ belong to the same class if $U(\sigma_b) = U(\sigma'_{b'})$ and we write $\sigma'_{b'} \sim \sigma_b$.

For any set A we denote by $|A|$ the number of elements in A.

Using a combinatorial calculations one can prove the following

Lemma 1. 1) *For any configuration σ_b we have*

$$U(\sigma_b) \in \{U_0, U_1, \ldots, U_{k+1}\},$$

where

$$U_i = \left(\frac{k+1}{2} - i\right)J_1 + \left(\frac{k(k+1)}{2} + 2i(i-k-1)\right)J_2, \quad i = 0, 1, \ldots, k+1. \tag{5}$$

2) *Let $\mathcal{C}_i = \Omega_i \cup \Omega_i^-$, $i = 0, \ldots, k+1$, where*

$$\Omega_i = \{\sigma_b : \sigma_b(c_b) = +1, |\{x \in b \setminus \{c_b\} : \sigma_b(x) = -1\}| = i\},$$

$$\Omega_i^- = \{-\sigma_b = \{-\sigma_b(x), x \in b\} : \sigma_b \in \Omega_i\},$$

and c_b is the center of the ball b. Then for $\sigma_b \in \mathcal{C}_i$ we have$U(\sigma_b) = U_i$.

3) *The class $\mathcal{C}_i$ contains $\frac{2(k+1)!}{i!(k-i+1)!}$ configurations.*

Lemma 2. *The relative Hamiltonian (3) has the form*

$$H(\sigma, \varphi) = \sum_{b \in M} (U(\sigma_b) - U(\varphi_b)). \tag{6}$$

Proof: Note that for any two vertices x and y such that $\langle x, y \rangle$ there exist exactly 2 unit balls $b, b' \in M$ such that $x, y \in b \cap b'$. Also, for any two vertices u and v such that $d(u, v) = 2$ there is a unique ball b such that $u, v \in b$. This completes the proof.

Theorem 3. For any class $\mathcal{C}_i$ and for any bounded configuration $\sigma_b \in \mathcal{C}_i$ there exists a periodic configuration φ with period non exceeding 2 such that $\varphi_{b'} \in \mathcal{C}_i$ for any $b' \in M$ and $\varphi_b = \sigma_b$.

Proof: For arbitrary given class $\mathcal{C}_i$ and $\sigma_b \in \mathcal{C}_i$ we shall construct configuration φ as follows. Without loss of generality we can take b as the ball with the center $e \in G_k$ (here e is the identity of G_k) i.e $b = \{e, a_1, \ldots, a_{k+1}\}$. Assume $\sigma_b(e) = +1$ (the case $\sigma_b(e) = -1$ is very similar). Denote $F = \{j \in \{1, \ldots, k+1\} : \sigma_b(a_j) = -1\}$. Note that $|F| = i$ since $\sigma_b(e) = +1$ and $\sigma_b \in \mathcal{C}_i$.

Consider two cases:

Case $i = 0$. In this case we have $\sigma_b(x) = 1$ for any $x \in b$, so configuration φ coincides with translational-invariant one $\varphi^+ = \{\varphi(x) \equiv +1\}$. Thus the period of φ is 1.

Case $i \geq 1$. Consider

$$\mathcal{H}_i = \left\{ x \in G_k : \sum_{j \in F} \omega_j(x) - \text{even} \right\},$$

where $\omega_j(x)$ is the number of a_j in $x \in G_k$. Note (see [8]) that $\mathcal{H}_i$ is normal subgroup of index 2 for G_k. By our construction (and assumption $\sigma_b(e) = +1$) we have $\sigma_b(x) = +1$ for any $x \in b \cap \mathcal{H}_i$ and $\sigma_b(u) = -1$ for any $u \in b \cap (G_k \setminus \mathcal{H}_i)$.□

We continue the bounded configuration $\sigma_b \in \mathcal{C}_i$ to whole lattice Γ^k (which we denote by φ) by

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in \mathcal{H}_i \\ -1 & \text{if } x \in G_k \setminus \mathcal{H}_i \end{cases}$$

So we obtain a periodic configuration φ with period 2 (=index of the subgroup); then by the construction $\varphi_b = \sigma_b$. Now we shall prove that all restrictions $\varphi_{b'}, b' \in M$ of the configuration φ belong to $\mathcal{C}_i$. Since $\mathcal{H}_i$ is the subgroup of index 2 in G_k, the quotient group has the form $G_k/\mathcal{H}_i = \{\mathcal{H}^0, \mathcal{H}^1\}$ with the cosets $\mathcal{H}^0 = \mathcal{H}_i, \mathcal{H}^1 = G_k \setminus \mathcal{H}_i$. Let $q_j(x) = |S_1(x) \cap \mathcal{H}^j|, j = 0, 1$; where $S_1(x) = \{y \in G_k : \langle x, y \rangle\}$, the set of all nearest neighbors of $x \in G_k$.

Denote $Q(x) = (q_0(x), q_1(x))$. Clearly, $q_0(x)$ $(resp. q_1(x))$ is the number of points y in $S_1(x)$ such that $\varphi(y) = +1$ $(resp. \varphi(y) = -1)$.

We note (see [18]) that for every $x \in G_k$ there is a permutation π_x of the coordinates of the vector $Q(e)$ (where e as before is the identity of G_k) such that

$$\pi_x Q(e) = Q(x).$$

Moreover $Q(x) = Q(e)$ if $x \in \mathcal{H}^0$ and $Q(x) = (q_1(e), q_0(e))$ if $x \in \mathcal{H}^1$. Thus for any $b' \in M$ we have (i) if $c_{b'} \in \mathcal{H}^0$ (where as before $c_{b'}$ is the center of b') then $\varphi_{b'} = \sigma_b$ up to a rotation; (ii) if $c_{b'} \in \mathcal{H}^1$ then $\varphi_{b'} = -\sigma_b$ up to a rotation. Since both $\sigma_b, -\sigma_b \in \mathcal{C}_i$ we get $\varphi_{b'} \in \mathcal{C}_i$ for any $b' \in M$. The theorem is proved. □

Definition 4. A configuration φ is called a *ground state* for the relative Hamiltonian H if

$$U(\varphi_b) = \min\{U_0, U_1, \ldots, U_{k+1}\}, \quad \text{for any } b \in M. \tag{7}$$

Remarks 1. *Usually, more simple and interesting ground states are periodic ones. In this paper we describe some non periodic ground states as well (cf.* [20] *chapter 2).*

2. A periodic ground state can be defined differently (see [20]) as a periodic configuration φ such that for any configuration σ that coincides with φ almost everywhere and $H(\varphi, \sigma) \leq 0$. It is easy to see that if φ is a ground state in the sense of Definition 4, then it satisfies $H(\varphi, \sigma) \leq 0$. In [10,16] it was proved that these two definitions (for periodic ground states) are equivalent for Hamiltonians on Z^d. But there is a problem to prove the equivalence of these definitions for Hamiltonians on the Cayley tree: normally the ratio of the number of boundary sites to the number of interior sites of a lattices becomes small in the thermodynamic limit of a large system. For the Cayley tree it does not, since both numbers grow exponentially like k^n. Correspondingly, we make a

Conjecture 1. The conditions (7) and $H(a, \sigma) \leq 0$ are equivalent.

We set

$$U_i(J) = U(\sigma_b, J), \quad \text{if } \sigma_b \in \mathcal{C}_i, \quad i = 0, 1, \dots, k+1.$$

The quantity $U_i(J)$ is a linear function of the parameter $J \in R^2$. For every fixed $m = 0, 1, \dots, k+1$ we denote

$$A_m = \{J \in R^2 : U_m(J) = \min\{U_0(J), U_1(J), \dots, U_{k+1}(J)\}\}. \tag{8}$$

It is easy to check that

$$A_0 = \{J \in R^2 : J_1 \leq 0; J_1 + 2kJ_2 \leq 0\};$$

$$A_m = \{J \in R^2 : J_2 \geq 0; 2(2m - k - 2)J_2 \leq J_1 \leq 2(2m - k)J_2\},$$

$$m = 1, 2, \dots, k; A_{k+1} = \{J \in R^2 : J_1 \geq 0; J_1 - 2kJ_2 \geq 0\}$$

and $R^2 = \cup_{i=0}^{k+1} A_i$.

For any $A_i, A_j, i \neq j$ we have

$$A_i \cap A_j = \begin{cases} \{J : J_1 = 2(2i - k)J_2, J_2 \geq 0\} & \text{if } j = i+1, i = 0, 1, \dots, k \\ (0, 0) & \text{if } 1 < |i - j| < k+1 \\ \{J : J_1 = 0, J_2 \leq 0\} & \text{if } |i - j| = k+1 \end{cases} \tag{9}$$

Denote

$$B = A_0 \cap A_{k+1}, \quad B_i = A_i \cap A_{i+1}, \quad i = 0, \dots, k.$$
$$\tilde{A}_0 = A_0 \setminus (B \cup B_0), \quad \tilde{A}_{k+1} = A_{k+1} \setminus (B \cup B_k),$$
$$\tilde{A}_i = A_i \setminus (B_{i-1} \cup B_i), \quad i = 1, \dots, k.$$

Fix $J \in R^2$ and denote

$$N_J(\sigma_b) = |\{j : \sigma_b \in \mathcal{C}_j\}|.$$

Using (9) one can prove

Lemma 5. *For any $b \in M$ and σ_b we have*

$$N_J(\sigma_b) = \begin{cases} k+2 & \text{if } J = (0,0) \\ 2 & \text{if } J \in \left(\left(\cup_{i=0}^{k} B_i\right) \cup B\right) \setminus \{(0,0)\}, \\ 1 & \text{otherwise} \end{cases}$$

Let $GS(H)$ be the set of all ground states of the relative Hamiltonian H (see (3)). For any $\sigma = \{\sigma(x), x \in V\} \in \Omega$ denote $\overline{\sigma} = -\sigma = \{-\sigma(x), x \in V\}$.

Theorem 6. (i) If $J = (0,0)$ then $GS(H) = \Omega$.

(ii) If $J \in \tilde{A}_i$, $i = 0, \dots, k+1$ then

$$GS(H) = \left\{\sigma^{(i)}, \overline{\sigma}^{(i)}\right\}.$$

(iii) If $J \in B_i \setminus \{(0,0)\}$, $i = 0, \dots, k$ then

$$GS(H) = \left\{\sigma^{(i)}, \overline{\sigma}^{(i)}, \sigma^{(i+1)}, \overline{\sigma}^{(i+1)}\right\} \cup S_i,$$

where S_i contains at least a countable subset of non periodic ground states.

(iv) If $J \in B \setminus \{(0,0)\}$, then

$$GS(H) = \left\{\sigma^{(0)}, \overline{\sigma}^{(0)}, \sigma^{(k+1)}, \overline{\sigma}^{(k+1)}\right\}.$$

Here $\sigma^{(i)}, \overline{\sigma}^{(i)}$, $i = 0, \dots, k+1$ are periodic ground states such that on any $b \in M$ the bounded configurations $\sigma_b^{(i)}, \overline{\sigma}_b^{(i)} \in \mathcal{C}_i$, i.e. $\sigma^{(0)}, \overline{\sigma}^{(0)}$ are translational - invariant and $\sigma^{(i)}, \overline{\sigma}^{(i)}$, $i = 1, \dots, k+1$ are periodic with period 2.

Proof: The assertion (i) is trivial. In each case (ii)-(iv) for a given configuration σ_b which makes $U(\sigma_b)$ minimal, by Theorem 3 one can construct the periodic ground states $\sigma^{(i)}, \overline{\sigma}^{(i)}$ (with period non exceeding two). For each case the exact number of such ground state coincides with the number of the configurations σ_b which make $U(\sigma_b)$ minimal. Thus it remains to prove the existence of the set S_i defined in the case (iii). If $J \in B_i \setminus \{(0,0)\}$ then the minimum points of $U(\sigma_b)$ belong to the classes $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ i.e. $\sigma_b^{(i)} = \{\sigma_b^{(i)}(x), x \in b\}$, $\overline{\sigma}_b^{(i)} = \{-\sigma_b^{(i)}(x), x \in b\}$ such that

$$\sigma_b^{(j)}(c_b) = +1, |\{x \in b \setminus \{c_b\} : \sigma_b^{(j)}(x) = -1\}| = j, \quad j = i, i+1, b \in M. \tag{10}$$

Thus any ground state $\varphi \in \Omega$ must satisfy

$$\varphi_b \in \left\{\sigma_b^{(i)}, \overline{\sigma}_b^{(i)}, \sigma_b^{(i+1)}, \overline{\sigma}_b^{(i+1)}\right\}, \quad b \in M. \tag{11}$$

Now we shall construct ground states $\varphi \in \Omega$ which satisfy (11).

Note that the configurations $\sigma_b^{(i)}$ and $\sigma_{b'}^{(i)}$ $(b, b' \in M)$ are the same up to a motion in G_k so we shall omit b. Thus configuration

$\sigma^{(i)}$ is the configuration such that on any unit ball $b \in M$ the condition (10) is satisfied.

Suppose two unit balls b and b' are neighbors, i.e. they have a common edge. We shall then say that the two bounded configurations σ_b and $\sigma'_{b'}$ are *compatible* if they coincide on the common edge of the balls b and b'. Denote by $\mathcal{B}(b)$ the set of all neighbor balls of b.

Denote $\tilde{\Omega}_i = \{\sigma^{(i)}, \overline{\sigma}^{(i)}, \sigma^{(i+1)}, \overline{\sigma}^{(i+1)}\}$. For any $\omega, \nu \in \tilde{\Omega}_i$ denote by $n(\omega, \nu) \equiv n_i(\omega, \nu)$ the number of possibilities to set up the configuration ν as a compatible configuration (with ω) around (i.e on neighboring balls of the ball on which ω is given) the configuration ω. Clearly $n(\omega, \nu) \in \{0, 1, \dots, k+1\}$, for any $\omega, \nu \in \tilde{\Omega}_i$ $i = 0, \dots, k+1$.

Denote

$$\mathbf{N}_i \equiv \mathbf{N}_i^{(k)} =$$
$$\begin{pmatrix} n(\sigma^{(i)}, \sigma^{(i)}) & n(\sigma^{(i)}, \overline{\sigma}^{(i)}) & n(\sigma^{(i)}, \sigma^{(i+1)}) & n(\sigma^{(i)}, \overline{\sigma}^{(i+1)}) \\ n(\overline{\sigma}^{(i)}, \sigma^{(i)}) & n(\overline{\sigma}^{(i)}, \overline{\sigma}^{(i)}) & n(\overline{\sigma}^{(i)}, \sigma^{(i+1)}) & n(\overline{\sigma}^{(i)}, \overline{\sigma}^{(i+1)}) \\ n(\sigma^{(i+1)}, \sigma^{(i)}) & n(\sigma^{(i+1)}, \overline{\sigma}^{(i)}) & n(\sigma^{(i+1)}, \sigma^{(i+1)}) & n(\sigma^{(i+1)}, \overline{\sigma}^{(i+1)}) \\ n(\overline{\sigma}^{(i+1)}, \sigma^{(i)}) & n(\overline{\sigma}^{(i+1)}, \overline{\sigma}^{(i)}) & n(\overline{\sigma}^{(i+1)}, \sigma^{(i+1)}) & n(\overline{\sigma}^{(i+1)}, \overline{\sigma}^{(i+1)}) \end{pmatrix}.$$

It is easy to see that

$$\mathbf{N}_0 = \begin{pmatrix} k+1 & 0 & k+1 & 0 \\ 0 & k+1 & 0 & k+1 \\ k & 0 & k & 1 \\ 0 & k & 1 & k \end{pmatrix}, \quad \mathbf{N}_i = \begin{pmatrix} k-i+1 & i & k-i+1 & i \\ i & k-i+1 & i & k-i+1 \\ k-i & i+1 & k-i & i+1 \\ i+1 & k-i & i+1 & k-i \end{pmatrix},$$

$i = 1, \dots, k-1.$

$$\mathbf{N}_k = \begin{pmatrix} 1 & k & 0 & k \\ k & 1 & k & 0 \\ 0 & k+1 & 0 & k+1 \\ k+1 & 0 & k+1 & 0 \end{pmatrix}, \quad \mathbf{N}_{k+1} = \begin{pmatrix} k+1 & 0 & 0 & 0 \\ 0 & k+1 & 0 & 0 \\ 0 & 0 & 0 & k+1 \\ 0 & 0 & k+1 & 0 \end{pmatrix}.$$

Consider $k+1$ sets $\mathbf{Q}_i = \{Q\}, i = 0, \dots, k$ of matrices $Q = \{q(u, v)\}_{u,v \in \tilde{\Omega}_i}$ such that

$$q(u, v) \in \{0, 1, \dots, n(u, v)\}, \sum_{v \in \tilde{\Omega}_i} q(u, v) = k+1, \forall u \in \tilde{\Omega}_i,$$

$q(u, \sigma^{(i)}) + q(u, \sigma^{(i+1)}) = n(u, \sigma^{(i)}),$ $q(u, \overline{\sigma}^{(i)}) + q(u, \overline{\sigma}^{(i+1)}) = n(u, \overline{\sigma}^{(i)}), i = 0, \dots, k$ and $q(u, v) = 0$ if and only if $q(v, u) = 0, u, v \in \tilde{\Omega}_i$.

Using matrices $\mathbf{N}_i$ we have

$$\mathbf{Q}_0 = \left\{ Q = \begin{pmatrix} a & 0 & k-a+1 & 0 \\ 0 & b & 0 & k-b+1 \\ c & 0 & k-c & 1 \\ 0 & d & 1 & k-d \end{pmatrix} \right\},$$

where $a, b \in \{0, 1, \ldots, k+1\}; c, d \in \{0, 1, \ldots, k\}; a = k+1$ iff $c = 0; b = k+1$ iff $d = 0$.

For $i = 1, \ldots, k-1$ we have

$$\mathbf{Q}_i = \left\{ Q = \begin{pmatrix} a_1 & b_1 & k-i-a_1+1 & i-b_1 \\ b_2 & a_2 & i-b_2 & k-i-a_2+1 \\ a_3 & b_3 & k-i-a_3 & i-b_3+1 \\ b_4 & a_4 & i-b_4+1 & k-i-a_4 \end{pmatrix} \right\},$$

where $a_1, \ a_2 \in \{0, 1, \ldots, k-i+1\}; \ a_3, \ a_4 \in \{0, 1, \ldots, k-i\}; \ b_1, b_2 \in \{0, \ldots, i\}; b_3, b_4 \in \{0, \ldots, i+1\}; a_1 = k-i+1$ iff $a_3 = 0; \ a_2 = k-i+1$ iff $a_4 = 0; b_1 = 0$ iff $b_2 = 0; b_1 = i$ iff $b_4 = 0; b_2 = i$ iff $b_3 = 0; b_3 = i+1$ iff $b_4 = i+1$.

For $i = k$ we have

$$\mathbf{Q}_k = \left\{ Q = \begin{pmatrix} 1 & a & 0 & k-a \\ b & 1 & k-b & 0 \\ 0 & c & 0 & k-c+1 \\ d & 0 & k-d+1 & 0 \end{pmatrix} \right\},$$

here $a, b \in \{0, 1, \ldots, k\}; c, d \in \{0, 1, \ldots, k+1\}; a = 0$ iff $b = 0; a = k$ iff $d = 0; b = k$ iff $c = 0; c = k+1$ iff $d = k+1$.

For a given $\xi \in \tilde{\Omega}_i$ and $Q = \{q(u, v)\}_{u,v \in \tilde{\Omega}_i} \in \mathbf{Q}_i$ we recurrently construct a ground state $\varphi^{Q,\xi}$ by the following way: fix a ball $b \in M$ and put on b the configuration $\varphi_b^{Q,\xi} := \xi$. On balls taken from $\mathcal{B}(b)$ we set exactly $q(\xi, \omega)$ copies of ω for any $\omega \in \tilde{\Omega}_i$. Thus configurations $\varphi_{b'}^{Q,\xi}, b' \in \mathcal{B}(b)$ are defined. Using these configurations, we define configurations on the balls $\mathcal{B}(b') \setminus \{b\}$, $(b' \in \mathcal{B}(b))$ putting $q(\varphi_{b'}^{Q,\xi}, \nu)$ copies of $\nu \in \tilde{\Omega}_i \setminus \{\xi\}$ and $q(\varphi_{b'}^{Q,\xi}, \xi) - 1$ copies of ξ which are compatible with $\varphi_{b'}^{Q,\xi}$. Further, on the balls $\mathcal{B}(b'') \setminus \{b'\}$, $(b'' \in \mathcal{B}(b'), b' \in \mathcal{B}(b))$ we set $q(\varphi_{b''}^{Q,\xi}, \epsilon)$ copies of $\epsilon \in \tilde{\Omega}_i \setminus \{\varphi_{b'}^{Q,\xi}\}$ and $q(\varphi_{b''}^{Q,\xi}, \varphi_{b'}^{Q,\xi}) - 1$ copies of $\varphi_{b'}^{Q,\xi}$ which are compatible with $\varphi_{b''}^{Q,\xi}$. Repeating this construction one can obtain a ground state $\varphi^{Q,\xi}$ such that

$$\varphi_b^{Q,\xi} \in \tilde{\Omega}_i, |\{b' \in \mathcal{B}(b) : \varphi_b^{Q,\xi} = \omega, \varphi_{b'}^{Q,\xi} = \nu\}| = q(\omega, \nu),$$

for any $b \in M$ and $\omega, \nu \in \tilde{\Omega}_i$.

In general the ground state $\varphi^{Q,\xi}$ is non periodic (see example below). It is easy to see that

$$\varphi^{Q^{(i)},\sigma^{(j)}} \equiv \sigma^{(j)}, \quad \varphi^{Q^{(i)},\overline{\sigma}^{(j)}} \equiv \overline{\sigma}^{(j)}, j = i, i+1, \quad i = 0, \ldots, k,$$

where

$$Q^{(i)} = \begin{pmatrix} k-i+1 & i & 0 & 0 \\ i & k-i+1 & 0 & 0 \\ 0 & 0 & k-i & i+1 \\ 0 & 0 & i+1 & k-i \end{pmatrix}. \tag{12}$$

Now using the ground states $\varphi^{Q,\xi}$ we shall construct an infinite set of ground states by the following way: one can choose $\xi \neq \eta, \xi, \eta \in \tilde{\Omega}_i$ and $Q_1, Q_2 \in \mathbf{Q}_i$ such that for configurations $\varphi^{Q_1,\xi}, \varphi^{Q_2,\eta}$ there are infinitely many $b \in M$ on which $\varphi_b^{Q_1,\xi}$ and $\varphi_{b'}^{Q_2,\eta}$ are compatible for some $b' \in \mathcal{B}(b)$. Indeed it is enough to take $\xi \neq \eta$ such that $q_1(\xi, \eta)q_2(\xi, \eta) \neq 0$ (see example below).

Denote

$$M_1 \equiv M_1^{\xi\eta}(Q_1, Q_2) = \{b \in M : \varphi_b^{Q_1,\xi}$$

is compatible with $\varphi_{b'}^{Q_2,\eta}$ for some $b' \in \mathcal{B}(b)\}$;

$$\mathcal{N}_1 = \{n \in \{0, 1, ...\} : \exists b \in M_1 \text{such that } |c_b| = n\};$$
$$V^{(y)} = \{z \in V : y < z\}.$$

Fix $m \in \mathcal{N}_1$ and denote

$$\tilde{W}_m = \{x \in W_m : \exists b \in M_1 \text{such that } c_b = x\}.$$

Consider the configuration

$$\varphi_m^{Q_1,Q_2,\xi,\eta}(x) = \begin{cases} \varphi^{Q_1,\xi}(x) & \text{if } x \in V_m \cup \{V^{(y)}, y \in W_m \setminus \tilde{W}_m\} \\ \varphi^{Q_2,\eta}(x) & \text{if } x \in V^{(y)}, y \in \tilde{W}_m. \end{cases}$$

Clearly $\varphi_m^{Q_1,Q_2,\xi,\eta}$, $m \in \mathcal{N}_1$ is a ground state and the number of such ground states is infinite, since $|\mathcal{N}_1| = \infty$. This completes the proof of the assertion (iii). The theorem is proved. □

Remark. *The proof of (ii) and (iv) can be obtained by using the above matrices. Indeed in case (i) $\tilde{\Omega}_i$ contains just $\sigma^{(i)}$ and $\overline{\sigma}^{(i)}$. Thus $\mathbf{Q}_i$ contains just matrices of type $Q^{(i)}$ (see (12)). In case (iv) $\tilde{\Omega}_{k+1} = \{\sigma^{(0)}, \overline{\sigma}^{(0)}, \sigma^{(k+1)}, \overline{\sigma}^{(k+1)}\}$ and $\mathbf{Q}_{k+1}$ contains the unique matrix*

$$Q_{k+1} = \begin{pmatrix} k+1 & 0 & 0 & 0 \\ 0 & k+1 & 0 & 0 \\ 0 & 0 & 0 & k+1 \\ 0 & 0 & k+1 & 0 \end{pmatrix}.$$

Consequently,

$$\varphi^{Q_{k+1},\xi} = \begin{cases} \varphi^{+} & \text{if } \xi = \sigma^{(0)} \\ \varphi^{-} & \text{if } \xi = \overline{\sigma}^{(0)} \\ \varphi^{\pm} & \text{if } \xi = \sigma^{(k+1)} \\ \varphi^{\mp} & \text{if } \xi = \overline{\sigma}^{(k+1)} \end{cases}$$

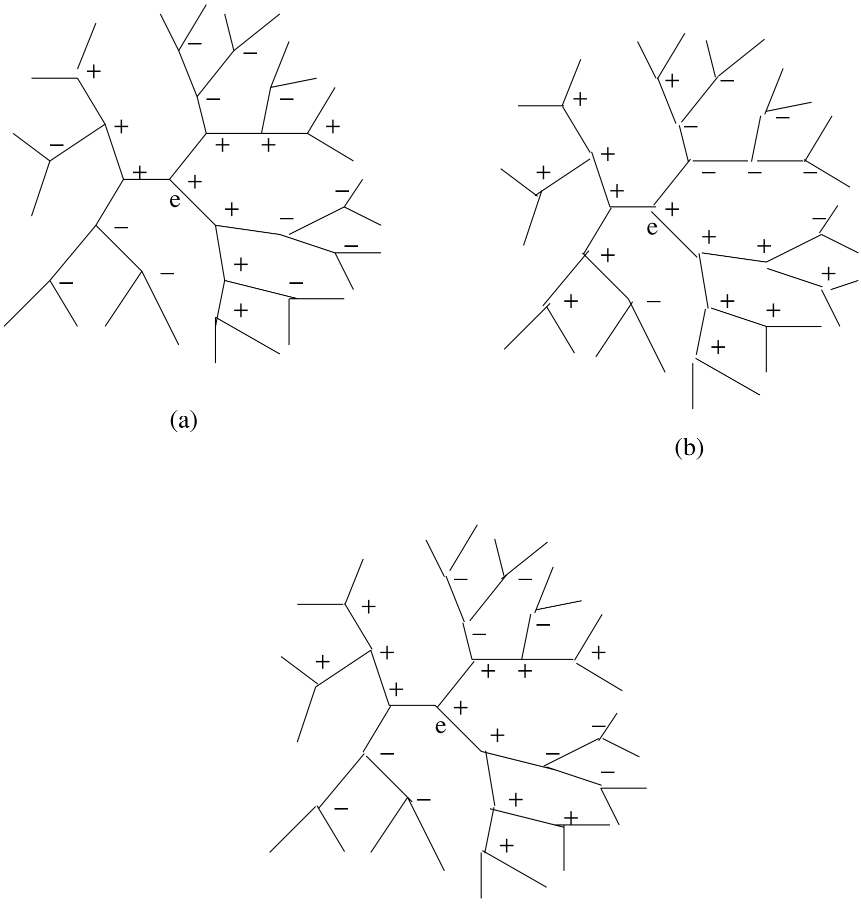
Here $\varphi^{\epsilon} = \{\varphi(x) \equiv \epsilon\}, \epsilon = +1, -1$ is translational-invariant which coincides with either $\sigma^{(0)}$ or $\overline{\sigma}^{(0)}$. The configuration $\varphi^{\pm} = -\varphi^{\mp}$ is periodic with respect to the subgroup $G_k^{(2)} = \{x \in G_k : |x| - \text{even}\} \subset G_k$ (chess-board) and coincides with $\sigma^{(k+1)} = -\overline{\sigma}^{(k+1)}$.

Example. Consider $k = 2, i = 0, J \in B_0 \setminus \{(0, 0)\}$. Take matrices

$$Q_1 = \begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

and $\xi = \sigma^{(0)}, \eta = \sigma^{(1)}$. The configurations $\varphi^{Q_1,\xi}$, $\varphi^{Q_2,\eta}$ and $\varphi_2^{Q_1,Q_2,\xi,\eta}$ are represented in figures 1(a), (b), and (c) respectively.

Remark. *Note that the way of the description of an infinite number of ground states used in the proof of (iii) is not a unique. One can use $\varphi^{Q,\xi}$ for another way to describe another infinite set of ground states.*



(a) (b) (c)

Fig. 1. Ground states.

4. THE PEIERLS CONDITION

Definition 7. Let $GS(H)$ be the complete set of all ground states of the relative Hamiltonian H. A ball $b \in M$ is said to be an *improper* ball of the configuration σ if $\sigma_b \neq \varphi_b$ for any $\varphi \in GS(H)$. The union of the improper balls of a configuration σ is called the *boundary* of the configuration and denoted by $\partial(\sigma)$.

Definition 8. The relative Hamiltonian H with the set of ground states $GS(H)$ satisfies the Peierls condition if for any $\varphi \in GS(H)$ and any configuration σ

coinciding almost everywhere with φ,

$$H(\sigma, \varphi) \geq \lambda |\partial(\sigma)|,$$

where λ is a positive constant which does not depend on σ, and $|\partial(\sigma)|$ is the number of unit balls in $\partial(\sigma)$.

Theorem 9. If $J \neq (0, 0)$ then the Peierls condition is satisfied.

Proof: Denote $\mathbf{U} = \{U_0, \ldots, U_{k+1}\}$ (see (5)), $U^{\min} = \min\{U_0, \ldots, U_{k+1}\}$ and

$$\lambda_0 = \min\{\mathbf{U} \setminus \{U_j : U_j = U^{\min}\}\} - U^{\min}. \tag{13}$$

Note that $U_0 = ... = U_{k+1}$ if and only if $J = (0, 0)$, consequently $\lambda_0 > 0$ if $J \neq (0, 0)$.

Suppose σ coincides almost everywhere with a ground state $\varphi \in GS(H)$ then we have $U(\sigma_b) - U(\varphi_b) \geq \lambda_0$ for any $b \in \partial(\sigma)$ since φ is a ground state. Thus

$$H(\sigma, \varphi) = \sum_{b \in M} (U(\sigma_b) - U(\varphi_b)) = \sum_{b \in \partial(\sigma)} (U(\sigma_b) - U(\varphi_b)) \geq \lambda_0 |\partial(\sigma)|.$$

Therefore, the Peierls condition is satisfied for $\lambda = \lambda_0$. The theorem is proved. $\square$

Remark. An interesting problem is to describe the set of Gibbs measures which corresponds to the set $GS(H)$. We shall study this problem in the next section. We expect that the structure of the set of periodic Gibbs measures is similar to the set of all periodic ground states i.e. there is no periodic Gibbs measure which corresponds to a non periodic ground state (cf. with the same problems in [6,12,20]). In the Section 5 for parameters J such that the model has only two periodic ground states we show that when temperature is low enough then there are two periodic Gibbs measures.

5. CONTOURS AND GIBBS MEASURES

Let $\Lambda \subset V$ be a finite set, $\Lambda' = V \setminus \Lambda$ and $\omega_\Lambda = \{\omega(x), x \in \Lambda'\}, \sigma_\Lambda = \{\sigma(x), x \in \Lambda\}$ be given configurations. The energy of the configuration σ_Λ has the form

$$H_\Lambda(\sigma|\omega_\Lambda) = J_1 \sum_{\substack{\langle x,y\rangle \\ x,y \in \Lambda}} \sigma(x)\sigma(y) + J_1 \sum_{\substack{\langle x,y\rangle \\ x \in \Lambda, y \in \Lambda'}} \sigma(x)\omega(y)$$
$$+ J_2 \sum_{\substack{x,y \in \Lambda \\ d(x,y)=2}} \sigma(x)\sigma(y) + J_2 \sum_{\substack{x \in \Lambda, y \in \Lambda' \\ d(x,y)=2}} \sigma(x)\omega(y). \tag{14}$$

Let $\omega^{\varepsilon}_{\Lambda'} \equiv \varepsilon, \varepsilon = \pm 1$ be a constant configuration outside Λ. For a given ε we extend the configuration σ_{Λ} inside Λ to the Cayley tree by the constant configuration and denote this configuration by $\sigma^{\varepsilon}_{\Lambda}$ and $\Omega^{\varepsilon}_{\Lambda}$ the set of all such configurations.

Now we describe a boundary of the configuration $\sigma^{\varepsilon}_{\Lambda}$. For the sake of simplicity we consider only case $J \in \tilde{A}_0$. In this case by Theorem 6 we have $GS(H) = \{\sigma^{(0)}, \overline{\sigma}^{(0)}\} = \{\sigma^+ \equiv +1, \sigma^- \equiv -1\}$. Fix +-boundary condition. Put $\sigma_n = \sigma^+_{V_n}$ and $\sigma_{n,b} = (\sigma_n)_b$. By Definition 7 the boundary of the configuration σ_n is

$$\partial \equiv \partial(\sigma_n) = \{b \in M_{n+2} : \sigma_{n,b} \neq \sigma^+_b \text{ or } \sigma^-_b\},$$

where $M_n = \{b \in M : b \cap V_n \neq \emptyset\}$.

The boundary ∂ contains of $2k+2$ parts

$$\partial^+_i = \{b \in M_{n+2} : \sigma_{n,b} \in \Omega_i\}, i = 1, 2, \ldots, k+1;$$

$$\partial^-_i = \{b \in M_{n+2} : \sigma_{n,b} \in \Omega^-_i\}, i = 1, 2, \ldots, k+1,$$

where Ω_i and Ω^-_i are defined in Lemma 1.

Consider V_n and for a given configuration σ_n (with "+"-boundary condition) denote

$$V^-_n \equiv V^-_n(\sigma_n) = \{t \in V_n : \sigma_n(t) = -1\}.$$

Let $G^n = (V^-_n, L^-_n)$ be the graph such that

$$L^-_n = \{l = \langle x, y\rangle \in L : x, y \in V^-_n\}.$$

It is clear, that for a fixed n the graph G^n contains a finite $(= m)$ of maximal connected subgraphs G^n_r i.e

$$G^n = \{G^n_1, \ldots, G^n_m\}, G^n_r = (V^-_{n,r}, L^-_{n,r}), r = 1, \ldots, m.$$

Here $V^-_{n,r}$ is the set of vertices and $L^-_{n,r}$ the set of edges of G^n_r.

Two edges $l_1, l_2 \in L$ are called nearest neighboring edges if $|i(l_1) \cap i(l_2)| = 1$, and we write $\langle l_1, l_2\rangle_1$.

For a given graph G denote by $V(G)-$ the set of vertexes and by $E(G)-$ the set of edges of G.

$$D_{\text{edge}}(K) = \{l_1 \in L \setminus E(K) : \exists l_2 \in E(K) \text{such that } \langle l_1, l_2\rangle_1\}$$

The (finite) sets $D_{\text{edge}}(G^n_r)$ are called *subcontours* of the boundary ∂. The set $V^-_{n,r}, r = 1, .., m$ is called the *interior*, $\text{Int}D_{\text{edge}}(G^n_r)$, of $D_{\text{edge}}(G^n_r)$. For any two subcontours T_1, T_2 the distance $\text{dist}(T_1, T_2)$ is defined by

$$\text{dist}(T_1, T_2) = \min_{\substack{x \in V(T_1)\\ y \in V(T_2)}} d(x, y),$$

where $d(x, y)$ is the distance between $x, y \in V$ (see Section 2.1).

Definition 10. The subcontours T_1, T_2 are called *adjacent* if $\text{dist}(T_1, T_2) \le 2$. A set of subcontours $\mathcal{A}$ is called connected if for any two subcontours $T_1, T_2 \in \mathcal{A}$ there is a collection of subcontours $T_1 = \tilde{T}_1, \tilde{T}_2, \dots, \tilde{T}_l = T_2$ in the set $\mathcal{A}$ such that for each $i = 1, \dots, l-1$ the subcontours $\tilde{T}_i$ and $\tilde{T}_{i+1}$ are adjacent.

Definition 11. Any maximal connected set (component) of subcontours is called *contour* of the set ∂.

The set of edges from a contour γ is denoted by $\text{supp}\gamma$.

Remark. *Note that Definition 11 of contours coincides with the Definition 2 of* [19]. *In* [19] *the quantity* $|\text{supp}\gamma|$ *plays very important role. But in the present paper instate of* $|\text{supp}\gamma|$ *we will use the number of improper (see Definition 7) balls of* γ.

For a given contour γ we put

$$\text{imp}_i^{\varepsilon}\gamma = \{b \in \partial_i^{\varepsilon} : b \cap \gamma \ne \emptyset\}, \quad \varepsilon = -1, 1; \quad i = 1, \dots, k+1;$$
$$\text{imp}^{\varepsilon}\gamma = \cup_{i=1}^{k+1}\text{imp}_i^{\varepsilon}\gamma, \; \text{imp}\gamma = \cup_{\varepsilon=\pm 1}\text{imp}^{\varepsilon}\gamma;$$
$$|\gamma| = |\text{imp}\gamma|, \;\; |\gamma_i^{\varepsilon}| = |\text{imp}_i^{\varepsilon}\gamma|, \;\; |\gamma_i| = |\gamma_i^{+}| + |\gamma_i^{-}|.$$

It is easy to see that the collection of contours $\alpha = \{\gamma_r\}$ generated by the boundary σ_n has the following properties

(i) Every contour $\gamma \in \alpha$ lies inside of the set V_{n+1};

(ii) For every two contours $\gamma_1, \gamma_2 \in \alpha$ we have dist $(\gamma_1, \gamma_2) > 2$, thus their supports $\text{supp}\gamma_1$ and $\text{supp}\gamma_2$ are disjoint.

A collection of contours $\alpha = \{\gamma\}$ that has the properties (i)-(ii) is called a configuration of contours. As we have seen, the configuration σ_n of spin generates the configuration of contours $\alpha = \alpha(\sigma_n)$. The converse assertion is also true. Indeed, for a given collection of contours $\{\gamma_r\}_{r=1}^{m}$ we put $\sigma_n(x) = -1$ for each $x \in \text{Int}\gamma_r, r = 1, \dots, m$ and $\sigma_n(x) = +1$ for each $x \in V_n \setminus \cup_{r=1}^{m}\text{Int}\gamma_r$.

Let us define a graph structure on M (i.e. on the set of all unit balls of the Cayley tree) as follows. Two balls $b, b' \in M$ are connected by an edge if they are neighbors i.e have a common edge. Denote this graph by $G(M)$. Note that the graph $G(M)$ is a Cayley tree of order $k \ge 1$. Here the vertices of this graph are balls of M. Thus Lemma 1.2 of [5] can be reformulated as following

Lemma 13. *Let* $\tilde{N}_{n,G}(x)$ *be the number of connected subgraphs* $G' \subset G(M)$ *with* $x \in V(G')$ *and* $|V(G')| = n$. *Then*

$$\tilde{N}_{n,G}(x) \le (ek)^n.$$

For $A \subset V$ denote

$$B(A) = \{b \in M : b \subset A\};$$
$$D(A) = \{x \in V \setminus A : \exists y \in A,\ \text{such that} \langle x, y\rangle\};$$
$$D_{\text{int}}(A) = \{x \in A : \exists y \in V \setminus A,\ \text{such that } \langle x, y\rangle\}.$$

Using the induction over n one can prove

Lemma 14. *Let K be a connected subgraph of the Cayley tree Γ^2 of order two, such that $|V(K)| = n$, then $|D(V(K))| = n + 2$.*

For $x \in V$ we will write $x \in \gamma$ if $x \in V(\gamma)$.

Denote $N_r(x) = |\{\gamma : x \in \gamma, |\gamma| = r\}|$, where as before $|\gamma| = |\text{imp}\gamma|$.

Lemma 15 *(cf. with Lemma 6 in [19]).* *If $k = 2$ (i.e. the Cayley tree of order two). Then*

$$N_r(x) \le \text{Const} \cdot (4e)^{2r}. \tag{15}$$

Proof: Denote by K_γ the minimal connected subgraph of Γ^2, which contains a contour γ. It is easy to see that if $\gamma = \{\gamma_1, \ldots, \gamma_m\}$, $m \ge 1$, (where γ_i is subcontour) then

$$B(V(K_\gamma)) \subset \text{imp}\gamma \cup B(\text{Int}\gamma). \tag{16}$$

Note that $D(\text{Int}\gamma)$ as a set contains different points. So we have

$$|\gamma| = |D(\text{Int}\gamma)| + |D_{\text{int}}(\text{Int}\gamma)|;$$
$$|B(\text{Int}\gamma)| = |\text{Int}\gamma \setminus D_{\text{int}}(\text{Int}\gamma)| = |\text{Int}\gamma| - |D_{\text{int}}(\text{Int}\gamma)|.$$

Using Lemma 14 we have$|\text{Int}\gamma| = |D(\text{Int}\gamma)| - 2. Consequently,$

$$\begin{aligned}|B(\text{Int}\gamma)| &= |D(\text{Int}\gamma)| - |D_{\text{int}}(\text{Int}\gamma)| - 2\\ &= |\gamma| - 2|D_{\text{int}}(\text{Int}\gamma)| - 2.\end{aligned}$$

Thus from (16) we have

$$|B(V(K_\gamma))| \le 2(|\gamma| - |D_{\text{int}}(\text{Int}\gamma)| - 1).$$

Since γ contains m subcontours we have

$$|D_{\text{int}}(\text{Int}\gamma)| \ge m. \tag{17}$$

Hence we get from (17)

$$|B(V(K_\gamma))| \le 2(|\gamma| - m - 1).$$

Since $\gamma \subset K_\gamma$ we get $|\gamma| \le |B(K_\gamma)| \le 2(|\gamma| - m - 1)$. Hence $|\gamma| \ge 2m + 2$ which implies $1 \le m \le \frac{|\gamma|-2}{2}$. A combinatorial calculations show that

$$N_r(x) \le 4 \sum_{m=1}^{[r/2-1]} \binom{2r-2m-2}{r} \tilde{N}_{2r-2m-2,\Gamma^2}(b(x)), \tag{18}$$

where $[a]$ is the integer part of a and $b(x)$ is a ball b such that $x \in b$.

Using inequality $\binom{n}{r} \le 2^{n-1}, r \le n$ and Lemma 15 from (18) we get (15). The lemma is proved. □

Following lemma gives a contour representation of Hamiltonian

Lemma 16. *The energy $H_n(\sigma_n) \equiv H_{V_n}(\sigma_n|\omega_{V_n'} = +1)$ (see (14)) has the form*

$$H_n(\sigma_n) = \sum_{i=1}^{k+1}(U_i - U_0)|\partial_i| + |M_{n+2}|U_0, \tag{19}$$

where $|\partial_i| = |\partial_i^+| + |\partial_i^-|$.

Proof: Using equality $U(\sigma_b) = U(-\sigma_b)$ we have

$$H_n(\sigma_n) = \sum_{b\in M_{n+2}} U(\sigma_{n,b}) = \sum_{i=1}^{k+1} U_i|\partial_i| + (|M_{n+2}| - |\partial|)U_0. \tag{20}$$

Now using equality $|\partial| = \sum_{i=1}^{k+1}|\partial_i|$ from (20) we get (19). The lemma is proved. □

Lemma 17. *Assume $J \in \tilde{A}_0$. Let γ be a fixed contour and*

$$p_+(\gamma) = \frac{\sum_{\sigma_n:\gamma\in\partial} \exp\{-\beta H_n(\sigma_n)\}}{\sum_{\tilde{\sigma}_n} \exp\{-\beta H_n(\tilde{\sigma}_n)\}}.$$

Then

$$p_+(\gamma) \le \exp\{-\beta\lambda_0|\gamma|\}, \tag{21}$$

where λ_0 is defined by formula (13) and $\beta = \frac{1}{T}$, $T > 0-$ temperature.

Proof: Put $\Omega_\gamma = \{\sigma_n : \gamma \subset \partial\}$, $\Omega_\gamma^0 = \{\sigma_n : \gamma \cap \partial = \emptyset\}$ and define a map $\chi_\gamma : \Omega_\gamma \to \Omega_\gamma^0$ by

$$\chi_\gamma(\sigma_n)(x) = \begin{cases} +1 & \text{if } x \in \mathrm{Int}\gamma \\ \sigma_n(x) & \text{if } x \notin \mathrm{Int}\gamma \end{cases}$$

For a given γ the map χ_γ is one-to-one map. We need to the following

Lemma 18. *For any* $\sigma_n \in \Omega_{V_n}$ *and* $i = 1, \dots, k+1$ *we have*

$$|\partial_i(\sigma_n)| = |\partial_i(\chi_\gamma(\sigma_n))| + |\gamma_i|.$$

Proof: It is easy to see that the map χ_γ destroys the contour γ and all other contours are invariant with respect to χ_γ. This completes the proof. □

Now we shall continue the proof of Lemma 17. By Lemma 16 we have

$$p_+(\gamma) = \frac{\sum_{\sigma_n \in \Omega_\gamma} \exp\{-\beta \sum_{i=1}^{k+1}(U_i - U_0)|\partial_i(\sigma_n)|\}}{\sum_{\tilde{\sigma}_n} \exp\{-\beta \sum_{i=1}^{k+1}(U_i - U_0)|\partial_i(\tilde{\sigma}_n)|\}} \leq$$

$$\frac{\sum_{\sigma_n \in \Omega_\gamma} \exp\{-\beta \sum_{i=1}^{k+1}(U_i - U_0)|\partial_i(\sigma_n)|\}}{\sum_{\tilde{\sigma}_n \in \Omega_\gamma^0} \exp\{-\beta \sum_{i=1}^{k+1}(U_i - U_0)|\partial_i(\tilde{\sigma}_n)|\}} =$$

$$\frac{\sum_{\sigma_n \in \Omega_\gamma} \exp\{-\beta \sum_{i=1}^{k+1}(U_i - U_0)|\partial_i(\sigma_n)|\}}{\sum_{\tilde{\sigma}_n \in \Omega_\gamma} \exp\{-\beta \sum_{i=1}^{k+1}(U_i - U_0)|\partial_i(\chi_\gamma(\tilde{\sigma}_n))|\}} \tag{22}$$

Since $J \in \tilde{A}_0$ by Theorem 6 we have $GS(H) = \{\sigma^+, \sigma^-\}$ hence $U_i - U_0 \geq \lambda_0$ for any $i = 1, \dots, k+1$. Thus using this fact and Lemma 18 from (22) we get (21). The lemma is proved. □

Using Lemmas 15 and 17 by very similar argument of [19] one can prove

Theorem 19. If $J \in \tilde{A}_0$ then for all sufficiently large β there are at least two Gibbs measures for the model (2) on Cayley tree of order two.

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